

Superfluid-insulator transition in a moving system of interacting bosons

E. Altman, A. Polkovnikov, E. Demler, B. Halperin, and M. D. Lukin

Physics Department, Harvard University, Cambridge, MA 02138

(Dated: November 2, 2004)

We analyze stability of superfluid currents in a system of strongly interacting ultra-cold atoms in an optical lattice. We show that such a system undergoes a dynamic, irreversible phase transition at a critical phase gradient that depends on the interaction strength between atoms. At commensurate filling, the phase boundary continuously interpolates between the classical modulation instability of a weakly interacting condensate and the equilibrium quantum phase transition into a Mott insulator state at which the critical current vanishes. We argue that quantum fluctuations smear the transition boundary in low dimensional systems. Finally we discuss the implications to realistic experiments.

Systems of ultracold atoms provide a unique ground for studying the physics of strongly correlated many body systems in a controlled manner. This has been demonstrated recently in spectacular experiments [1] involving atomic condensates in optical lattices, in which a quantum phase transition from a superfluid (SF) to a Mott insulating state (IN) [2, 3] was observed. One particularly intriguing aspect of these systems is that they can be well isolated from the environment. This allows one to study quantum dynamics of strongly interacting particles very far from thermal equilibrium [4, 5, 6, 7, 8, 10, 12], and, in particular, to investigate phase transitions in such non-equilibrium systems.

In this Letter we analyze the many-body physics associated with a superfluid to insulator transition of strongly interacting bosons, in the presence of super current flow. We show that such a system, undergoes a dynamic phase transition at a critical momentum that depends on the interaction strength. For any non-vanishing initial current, this transition is discontinuous and is followed by irreversible loss of the current. This is in contrast with the equilibrium case, where the system undergoes a continuous reversible transition into the insulating state and back into the superfluid state due to the adiabatic change in lattice potential [1]. We argue that quantum fluctuations play an important role near the phase transition and generally smear the sharp stability boundary in systems of lower dimensionality [8, 9]. These effects can be probed in experiments analogous to that of Ref. [1].

The dynamics of a weakly interacting Bose condensate moving in a periodic optical potential has been recently a subject of extensive studies [6, 7]. In particular, it was experimentally demonstrated that the condensate undergoes a dynamical localization transition involving onset of chaos [10, 11]

when the superfluid velocity exceeds the critical value. Similarly to the equilibrium SF-IN phase transition, the localization transition for weakly interacting bosons is accompanied by loss of phase coherence, but contrary to the former, it is not reversible. The dynamic phase transition described here interpolates continuously between the classical modulation instability and the quantum phase transition into the Mott state, thereby establishing a natural connection between the two.

We stress that compared to the classical GP picture quantum effects manifest in two *different* ways: (i) reduction of the critical current with interaction due to mean-field depletion, and (ii) smearing of the transition due to tunneling of phase slips beyond the mean-field theory. While the former effect exists in all dimensions, the latter is efficient only in low-dimensional systems [13].

Our results can be understood qualitatively by considering the superfluid current I associated with a condensate moving within the lowest Bloch band on a discrete lattice: $I \sim \langle a_{i+1}^\dagger a_i - a_i^\dagger a_{i+1} \rangle \sim \rho \sin(p)$. Here a_i is a bosonic operator on the i_{th} lattice cite, p is the (quasi)momentum of the condensate measured in the units of inverse lattice constant and ρ is the superfluid density. In a weakly interacting condensate ρ is independent of p . Thus, the current increases with p up to a maximal value at $p_c = \pi/2$. Beyond this point, the effective mass changes sign and any further increase in p results in decrease of the current, rendering the supercurrent unstable [13]. We note that $p_c = \pi/2$ exactly coincides with the onset of the modulational instability within the Gross-Pitavetskii (GP) approach [6, 7]. Beyond the classical level the superfluid density is itself a function of the effective mass and thus also of p . In particular, the Mott transition in a stationary condensate occurs as ρ vanishes when the effective mass (inverse hopping

amplitude) becomes sufficiently large. Since the latter increases with the momentum, a strongly interacting superfluid will generally become unstable at $p_c < \pi/2$. In fact, p_c vanishes in the vicinity of the Mott insulator state, where a small increase of the effective mass is sufficient to destroy superfluidity. These conclusions are illustrated in Figure 1, which shows a mean-field stability phase diagram for a condensate with the momentum p as a function of the normalized atom interaction strength at commensurate filling.

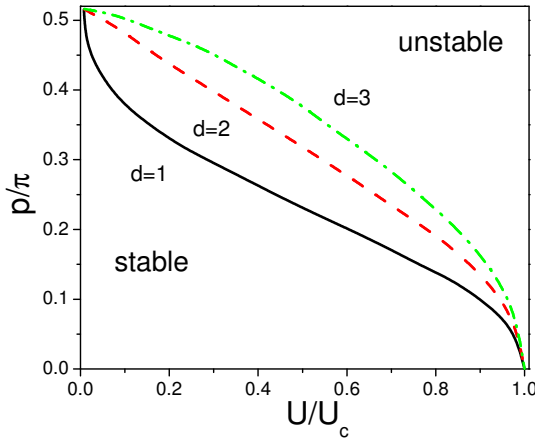


FIG. 1: Stability phase diagram in the plane of phase twist per bond versus dimensionless interaction for filling of $N = 1$ particle per site found from numerical solution of time dependent Gutzwiller equations (8).

Static and dynamic properties of condensates in optical lattices are described by the Bose Hubbard model (BHM)

$$H = \frac{U}{2} \sum_i n_i(n_i - 1) - J \sum_{\langle ij \rangle} (a_i^\dagger a_j + \text{H.c.}), \quad (1)$$

where J is the hopping amplitude between the nearest neighbors, U is the on-site repulsive interaction, and $n_i = a_i^\dagger a_i$ is the number operator. We denote the average number of bosons per site by N . Provided the system is deep in the superfluid phase ($JN \gg U$), the dynamics can be well approximated by the discrete GP equations

$$i \frac{d\psi_j}{dt} = -J \sum_{k \in O} \psi_k + U |\psi_j|^2 \psi_j, \quad (2)$$

where $\psi_i = \langle a_i \rangle$ is the matter field and the set O contains the nearest neighbors of site j . Linear

mode analysis around the stationary current carrying solutions $\psi_j = \sqrt{N} \exp(ipx_j)$, yields the onset of instability at $p = \pi/2$ [6, 7].

Close to the Mott transition, increased quantum fluctuations invalidate the GP description. However, one can still use semiclassical order parameter dynamics if one coarse grains the system into blocks of roughly a coherence length ξ , which is related to the superfluid density by a standard scaling form [15]. As in the weak coupling theory, the current should become unstable when the phase change per unit cell exceeds $\pi/2$, with the unit cell now of order ξ . Since the coherence length, ξ , diverges at the transition, we expect p_c to vanish as $1/\xi$, as we indeed find below. The diverging length scale facilitates a continuum description of the dynamics close to the Mott transition, in the form of a time dependent Ginzburg-Landau equation [12, 13]

$$\ddot{\psi} = \nabla^2 \psi + \psi(\xi^{-2} - |\psi|^2). \quad (3)$$

Within the mean-field approximation, and in the limit of large average occupation $N \gg 1$, $\xi^{-2} = 2d(1 - u)$, where $u = U/(8JNd)$ is the dimensionless interaction constant. If N is not too large then Eq.(3) still holds but the expressions for ξ and u are more complicated [13].

We choose the zero of energy at the state with integer filling. Then the deviation from commensurate density is given in terms of the superfluid order parameter ψ by $\delta n = C_d(\psi^* \dot{\psi} - \dot{\psi} \psi^*)/2i$, with $C_d = 1/(4d\sqrt{2d})$. Note that $\int d^d x \delta n$ is a constant of motion under (3). The supercurrent is similarly given by $I = C_d(\psi^* \nabla \psi - \psi \nabla \psi^*)/2i$.

The equation of motion (3) admits uniform solutions $\psi = \rho e^{ipx + i\mu t}$, with $\rho = \sqrt{\xi^{-2} + \mu^2 - p^2}$, which are characterized by a phase gradient p and a relative density

$$\delta n = C_d \mu (\xi^{-2} + \mu^2 - p^2). \quad (4)$$

In particular, at commensurate filling $\mu = 0$ and ψ is time-independent. To analyze whether these solutions are stable we find the spectrum of small fluctuations around them. There are two eigenmodes in the superfluid regime ($\xi^{-2} > 0$): a stable gapped mode, and a phase (Bogoliubov) mode with linear dispersion at long wave lengths. The spectrum of the latter, for wavevectors parallel to the current, reads:

$$\omega(k) = \frac{2\mu p}{2\mu^2 + \rho^2} k + \frac{\rho}{2\mu^2 + \rho^2} \sqrt{\xi^{-2} + 3\mu^2 - 3p^2} |k|. \quad (5)$$

Here the first term is analogous to the usual Doppler shift and the second describes propagation of the sound waves in the moving reference frame. The onset of imaginary frequencies marking the instability occurs at $p_c^2 - \mu(p_c)^2 = 1/3\xi^2$. Combining this with Eq. (4) we find that for $N \gg 1$

$$p_c = \sqrt{\left(\frac{3\delta n}{4N(1-u)}\right)^2 + \frac{2}{3}d(1-u)}. \quad (6)$$

In the case of commensurate filling $\delta n = 0$ the critical phase gradient p_c , decreases as $\sqrt{1-u}$ with increasing interaction, terminating at $p_c = 0$ at the equilibrium Mott transition ($u = 1$). As argued above this result can be understood in terms of the diverging coherence length, $p_c \sim 1/\xi \propto \sqrt{1-u}$. We note that for N of the order of one the explicit expression for the critical momentum p_c will be rather complicated but will preserve all the qualitative features of Eq. (6).

At incommensurate density there is no equilibrium Mott transition. As a result, we do not expect the instability to reach $p = 0$. Indeed, p_c has a minimum at $u < 1$ and diverges as $u \rightarrow 1$. The divergence, simply signals the breakdown of the continuum theory and is cutoff by the lattice.

To interpolate between the regimes of weak and strong interactions we employ the Gutzwiller approximation [14]. In this approach, the wavefunction is assumed to be factorizable:

$$|G\rangle = \prod_j \left[\sum_{n=0}^{\infty} f_{jn} |n\rangle_j \right]. \quad (7)$$

Here j is a site index and n is the site occupation. The ansatz (7) supplemented by self-consistency conditions leads to equations of motion for the variational parameters:

$$\begin{aligned} -i\dot{f}_{jn} &= \frac{U}{2}n(n-1)f_{jn} - \\ &- Jz(\sqrt{n}f_{j,n-1}\psi_j + \sqrt{n+1}f_{j,n+1}\psi_j^*), \end{aligned} \quad (8)$$

where

$$\psi_j \equiv \frac{1}{z} \sum_{i \in O} \langle G | a_i | G \rangle. \quad (9)$$

For actual calculations we truncated (7) at five and ten states per site without noticeable differences in the results.

Equations (8), admit uniform current carrying solutions. We numerically check their stability to

slight perturbations in the equations of motion. We show the stability boundaries at commensurate filling in Fig. 1. It is evident that the dynamical instability at $\pi/2$ in the GP regime is continuously connected to the equilibrium (zero current) Mott transition.

We perform a similar analysis at incommensurate filling (Fig. 2). In agreement with the continuum expression (6) we find that the critical momentum p_c reaches a minimum at some $u < 1$. At stronger interactions, p_c increases, and saturates at $\pi/2$.

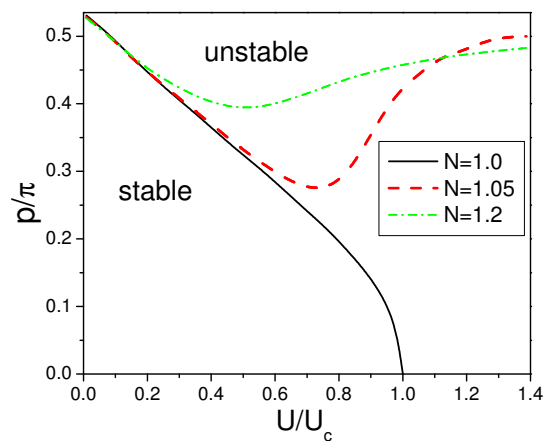


FIG. 2: Stability phase diagrams for different filling factors from numerical solution of the time dependent Gutzwiller equations (8). Away from commensurate filling the critical phase gradient reaches a minimum and climbs back to $\pi/2$.

The Gutzwiller approximation is equivalent to a time-dependent mean field theory. It thus ignores current decay that can occur below the mean field critical momentum due to quantum tunneling, and may lead to smearing of the transition. We describe in detail current decay due to quantum and thermal tunneling beyond mean-field theory in Ref. [13]. Here we summarize the results for the zero temperature system in the vicinity of the commensurate superfluid-insulator transition. The decay rate Γ_Q of a supercurrent can be calculated by means of the semiclassical instanton method [16]. Close to the instability $p \rightarrow p_c$ we can derive a universal expression for Γ_Q [13]:

$$\Gamma_Q \propto e^{-A_d(1-u)^{1/4}(p_c-p)^{2.5-d}}, \quad (10)$$

with A_d being a number of the order of one. In

three dimensions the scaling breaks down and Γ_Q is discontinuous at the transition, justifying the Mean field phase diagram. In one and two dimensions, there is a broad and smooth crossover, where the current decay rate continuously increases towards the meanfield phase boundary. In the one dimensional case this effect has been recently observed experimentally [17].

In realistic experimental situations condensates are confined in harmonic traps which leads to a non uniform density distribution in the form of domains with different N . In the weakly interacting regime the critical momentum is $p_c = \pi/2$, i.e. it is insensitive to the spacial density variation induced by the harmonic confinement. By contrast, in the regime of strong interactions the position of the dynamical instability strongly depends on the filling factor N that is directly affected by the density distribution. In particular, the motion first becomes unstable for the smallest integer filling $N = 1$.

In Fig. 3 we plot the time evolution of the condensate momentum (computed within the Gutzwiller approximation) in a trap, for two different filling factors. The center of mass motion becomes unstable at approximately the same interaction strength in both cases. But while at smaller filling the condensate motion rapidly becomes chaotic as in the uniform case, damping of oscillations at larger filling occurs much more gradually. These results can be understood by noting that if the phase gradient in the condensate exceeds the critical value corresponding to $N = 1$ these domains become unstable triggering the decay of current. However, when there is high filling of the central sites the overall weight of domains with $N = 1$, is small. Thus the effect of the instability on the total current is reduced.

An important experimental manifestation of these results is the inherently irreversible nature of the phase transition at finite currents. Consider a situation in which a moving condensate is first prepared on a weak lattice. Then, the depth of the periodic potential is increased adiabatically [18], which corresponds to moving along a horizontal line in the parameter space of Fig. (1). Finally, the lattice depth is slowly decreased back to its original state and the visibility of the interference fringes compared to their initial value. If in this sequence we pass the instability, then the current will decay into incoherent excitations and heat the condensate. This will result in total loss of current and reduced visibility of the interference fringes at

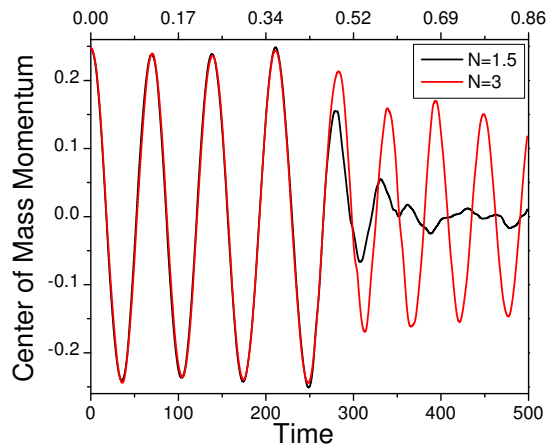


FIG. 3: Time dependence of the condensate momentum in a two-dimensional harmonic trap with different filling factors per central site. The simulated system is a lattice of dimensions 120×60 with global trapping potential $V(j_x, j_y) = 0.01(j_x^2 + j_y^2)$. We set the hopping amplitude $J = 1$ while increasing the interaction linearly in time: $U(t) = 0.01t$.

the end of the cycle. Such experiments could be used to probe the nonequilibrium phase diagram (Fig. 1) and to determine the position of the equilibrium Mott transition.

This work was supported by the NSF (grants DMR-01328074, DMR-0233773, DMR-0231631, DMR-0213805, PHY-0134776), the Sloan and the Packard Foundations, and by Harvard-MIT CUA.

-
- [1] M. Greiner, O. Mandel, T. Esslinger, T.W. Hänsch and I. Bloch, *Nature*, **415**, 39 (2002).
 - [2] M.P.A. Fisher, P.B. Weichman, G. Grinstein, D.S. Fisher, *Phys. Rev. B* **40**, 546 (1989).
 - [3] D. Jaksch, C. Bruder, J.I. Cirac, C.W. Gardiner, and P. Zoller, *Phys. Rev. Lett.* **81**, 3108 (1998).
 - [4] O. Mandel, M. Greiner, A. Widera, T. Rom, T.W. Hänsch, and I. Bloch, *Nature* **425**, 937 (2003).
 - [5] A. Polkovnikov, S. Sachdev, S.M. Girvin, *Phys. Rev. A* **66**, 053607 (2002).
 - [6] B. Wu and Q. Niu, *Phys. Rev. A*, **64** (2001) 061603(R)
 - [7] A. Smerzi, A. Trombettoni, P. G. Kevrekidis, and A. R. Bishop *Phys. Rev. Lett.* **89**, 170402 (2002)
 - [8] A. Polkovnikov, D.-W. Wang, *Phys. Rev. Lett.* **93**, 070401 (2004).

- [9] J. Gea-Banacloche, A.M. Rey, G. Pupillo, C.J. Williams, and C.W. Clark, cond-mat/0410677.
- [10] L. Fallani, L. De Sarlo, J.E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, cond-mat/0404045.
- [11] T. Anker, M. Albiez, R. Gati, S. Hunsmann, B. Eiermann, A. Trombettoni and M.K. Oberthaler, cond-mat/0410176.
- [12] E. Altman and A. Auerbach, Phys. Rev. Lett. 89, 250404 (2002)
- [13] A. Polkovnikov, E. Altman, E. Demler, B. Halperin and M. Lukin, to be published.
- [14] D. S. Rokhsar and B. G. Kotliar, Phys. Rev. B **44**, 10328 (1991).
- [15] S. Sachdev, "Quantum Phase Transitions", Cambridge
- [16] S. Coleman, Phys. Rev. D **15**, 2929 (1977); C.G. Callan and S. Coleman, *ibid* **16**, 1762 (1977).
- [17] C.D. Fertig, K.M. O'Hara, J.H. Huckans, S.L. Rolston, W.D. Phillips, and J.V. Porto, cond-mat/0410491.
- [18] One should be aware that if in a harmonic trap the tunneling (not the interaction strength) decrease in time then the amplitude of momentum oscillations increases (see Ref. [13] for the details).